ÇANKAYA UNIVERSITY

Department of Mathematics

MATH 156 - Calculus for Engineering II

SECOND MIDTERM EXAMINATION

05.12.2019

STUDENT NUMBER:

NAME-SURNAME:

SIGNATURE:

DURATION: 100 minutes

Question	Grade	Out of
1		25
2		25
3		25
4		25
Total		100

IMPORTANT NOTES:

1) Please make sure that you have written your student number and name above.

2) Check that the exam paper contains 4 problems.

3) Show all your work. No points will be given to correct answers without reasonable work.

Question 1. Let $\vec{u}$ and $\vec{v}$ be two vectors.

a) Show that $|\vec{u}.\vec{v}| \leq |\vec{u}||\vec{v}|$.

$$\vec{u}.\vec{v} = |\vec{u}||\vec{v}|\cos\theta$$

$$\Rightarrow |\vec{u}.\vec{v}| = |\vec{u}||\vec{v}||\cos\theta| \leq |\vec{u}||\vec{v}| \quad \text{as} \quad |\cos\theta| \leq 1$$

b) If $\vec{u}.\vec{v} = k$ and $\vec{u} \times \vec{v} = \left\langle \frac{k}{3}, \frac{k}{3}, \frac{k}{3} \right\rangle$ find the angle between $\vec{u}$ and $\vec{v}$, where k is a positive number.

$$\left.\begin{aligned} k &= \vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta \\ \frac{k}{\sqrt{3}} = \sqrt{\frac{k^2}{9}+\frac{k^2}{9}+\frac{k^2}{9}} &= |\vec{u}\times\vec{v}| = |\vec{u}||\vec{v}|\sin\theta \end{aligned}\right\} \quad \tan\theta = \frac{1}{\sqrt{3}} \Rightarrow \theta = \frac{\pi}{6}$$

Question 2. a) Let ℓ be a line of intersection of the planes $2x + y + z = 2$ and $x - 2y + 2z = 1$. Find an equation for ℓ.

$\vec{n}_1 = \langle 2,1,1 \rangle$

$\vec{n}_2 = \langle 1,-2,2 \rangle$

$$\vec{v} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 1 \\ 1 & -2 & 2 \end{vmatrix} = \vec{i}(2+2) - \vec{j}(4-1) + \vec{k}(-4-1) = \langle 4,-3,-5 \rangle$$

For $y=0$, $\left.\begin{array}{l} 2x+z=2 \\ -2/\ x+2z=1 \end{array}\right\}$ $-3z=0$ $z=0 \Rightarrow x=1$

Therefore $(1,0,0)$ lies on the intersection line of the planes.

$$\langle x,y,z \rangle = \langle 1,0,0 \rangle + t\langle 4,-3,-5 \rangle$$

$$\left.\begin{array}{l} x = 1+4t \\ y = -3t \\ z = -5t \end{array}\right\} t \in \mathbb{R}$$

b) Let

$$\ell_1 : \begin{array}{l} x = 1 + 3t, \\ y = -2 - 2t, \\ z = 3 + 4t, \end{array} \quad \ell_2 : \begin{array}{l} x = -2 + s, \\ y = 1 - s, \\ z = 4 + 2s, \end{array}$$

where $t, s \in \mathbb{R}$. Find an equation for the plane that contains ℓ_1 and parallel to ℓ_2.

$$\vec{n} = \vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -2 & 4 \\ 1 & -1 & 2 \end{vmatrix} = \vec{i}(-4+4) - \vec{j}(6-4) + \vec{k}(-3+2) = \langle 0,-2,-1 \rangle$$

The plane eq. is

$$\langle x-1, y+2, z-3 \rangle \cdot \langle 0,-2,-1 \rangle = 0$$

$$-2(y+2) - (z-3) = 0$$

$$2y + z = -1$$

Question 3. a) Find the limit $\lim_{(x,y)\to(0,0)} \arctan\left(\dfrac{y^2+x^3y^2+x^4}{x^4+y^2}\right)$ if it exists.

Let $f(x,y) = \dfrac{y^2+x^3y^2+x^4}{x^4+y^2}$. For $x=0$, $f(0,y)=1 \to 1$ as $(x,y)\to(0,0)$

For $y=0$, $f(x,0)=1 \to 1$ as $(x,y)\to(0,0)$

Let $L=1$ (?).

$$0 \le \left| \frac{y^2+x^3y^2+x^4}{x^4+y^2} - 1 \right| = \left| \frac{\cancel{y^2}+x^3y^2+\cancel{x^4}-\cancel{x^4}-\cancel{y^2}}{x^4+y^2} \right| = \left| \frac{x^3y^2}{x^4+y^2} \right| \le |x|^3 \to 0 \text{ as } (x,y)\to(0,0)$$

By sandwich theorem we get that $\lim_{(x,y)\to(0,0)} f(x,y) = 1$ and since $\arctan x$ is continuous at $x=1$ we have

$$\lim_{(x,y)\to(0,0)} \arctan\left(\frac{y^2+x^3y^2+x^4}{x^4+y^2}\right) = \arctan 1 = \frac{\pi}{4}$$

b) Is

$$f(x,y) = \begin{cases} 0, & y \le 0 \text{ or } y \ge x^4, \\ 1, & 0 < y < x^4, \end{cases}$$

continuous at $(0,0)$? Explain.

For $y \le 0$, $f(x,y) = 0 \to 0$ as $(x,y)\to(0,0)$

For $0<y<x^4$, $f(x,y) = 1 \to 1$ as $(x,y)\to(0,0)$

By two path test $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist and hence $f(x,y)$ is not continuous at $(0,0)$.

Question 4. a) Consider the function $z = z(u, v)$, where $u = x + t$, $v = x - t$. Show that $z_{xx} + z_{tt} = 2(z_{uu} + z_{vv})$.

$$z_x = z_u \cdot u_x + z_v \cdot v_x = z_u + z_v \qquad (u_x = 1,\ u_t = 1,\ v_x = 1,\ v_t = -1)$$

$$z_{xx} = z_{uu}u_x + z_{uv}v_x + z_{vu}u_x + z_{vv}v_x = z_{uu} + z_{uv} + z_{vu} + z_{vv}$$

$$z_t = z_u \cdot u_t + z_v v_t = z_u - z_v$$

$$z_{tt} = z_{uu}u_t + z_{uv}v_t - z_{vu}u_t - z_{vv}v_t = z_{uu} - z_{uv} - z_{vu} + z_{vv}$$

$$z_{xx} + z_{tt} = 2(z_{uu} + z_{vv})$$

b) Find the direction at which the function $f(x, y)$ has the maximum rate of change at the point $P(2, -5)$ where $f(x, y)$ is differentiable at P, $\vec{u} = \left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle$ and $\vec{w} = \left\langle -\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right\rangle$ such that $(D_{\vec{u}}f)(P) = 4\sqrt{5}$ and $(D_{\vec{w}}f)(P) = -6\sqrt{5}$.

Let $\vec{\nabla} f(2,-5) = \langle a, b \rangle$

$$(D_{\vec{u}}f)(P) = \langle a, b\rangle \cdot \left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle = 4\sqrt{5} \qquad \frac{a}{\sqrt{5}} + \frac{2b}{\sqrt{5}} = 4\sqrt{5}$$

$$(D_{\vec{w}}f)(P) = \langle a, b\rangle \cdot \left\langle -\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right\rangle = -6\sqrt{5} \qquad -\frac{2a}{\sqrt{5}} + \frac{b}{\sqrt{5}} = -6\sqrt{5}$$

$$2 / \quad \begin{cases} a + 2b = 20 \\ -2a + b = -30 \end{cases} \quad 5b = 10 \Rightarrow b = 2,\ a = 16 \Rightarrow \vec{\nabla} f(2,-5) = \langle 16, 2 \rangle$$

So the direction at which $f(x,y)$ has maximum rate of change at P is the direction of $\vec{\nabla} f(2,-5) = \langle 16, 2 \rangle$