ÇANKAYA UNIVERSITY

Department of Mathematics

MATH 156 - Calculus for Engineering II

FIRST MIDTERM EXAMINATION
31.10.2019

Question	Grade	Out of
1		25
2		25
3		25
4		25
Total		100

STUDENT NUMBER:

NAME-SURNAME:

SIGNATURE:

DURATION: 100 minutes

IMPORTANT NOTES:

1) Please make sure that you have written your student number and name above.

2) Check that the exam paper contains 4 problems.

3) Show all your work. No points will be given to correct answers without reasonable work.

Question 1. Decide whether the given sequences are convergent or divergent. Explain.

a) $\{a_n\}_{n=4}^{\infty} = \left\{ \frac{1}{\sqrt{15}-\sqrt{20}}, \frac{1}{\sqrt{24}-\sqrt{30}}, \ldots, \frac{1}{\sqrt{n^2-1}-\sqrt{n^2+n}}, \ldots \right\}$

$$\lim_{n\to\infty} \frac{1}{\sqrt{n^2-1}-\sqrt{n^2+n}} = \lim_{n\to\infty} \frac{1}{\sqrt{n^2-1}-\sqrt{n^2+n}} \cdot \frac{\sqrt{n^2-1}+\sqrt{n^2+n}}{\sqrt{n^2-1}+\sqrt{n^2+n}}$$

$$= \lim_{n\to\infty} \frac{\sqrt{n^2-1}+\sqrt{n^2+n}}{\cancel{n^2}-1-\cancel{n^2}-n} = \lim_{n\to\infty} \frac{n}{n} \frac{\sqrt{1-\frac{1}{n^2}}+\sqrt{1+\frac{1}{n}}}{-\frac{1}{n}-1} = \frac{2}{-1} = -2$$

So $\{a_n\}$ is convergent.

b) $\{b_n\}_{n=2}^{\infty} = \left\{ \frac{1}{2}\int_1^2 \frac{1}{x}dx, \frac{1}{3}\int_1^3 \frac{1}{x}dx, \ldots, \frac{1}{n}\int_1^n \frac{1}{x}dx, , \ldots \right\}$

$$\lim_{n\to\infty} \frac{1}{n}\int_1^n \frac{1}{x}dx = \lim_{n\to\infty} \frac{1}{n}\ln n = \lim_{x\to\infty} \frac{\ln x}{x} \overset{L'H}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$$

So $\{b_n\}$ is convergent

Question 2. a) Give an example for the case that $\sum_{n=1}^{\infty} a_n$ is convergent but $\sum_{n=1}^{\infty} \sqrt{a_n}$ is divergent, where $a_n > 0$ for all $n = 1, 2, ...$

Let $a_n = \frac{1}{n^2}$. Then $a_n > 0$ for all $n = 1, 2, \ldots$.

Moreover $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent (p-series with $p = 2 > 1$) and

$\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent (harmonic series or p-series with $p = 1 \leq 1$)

b) Is $\sum_{n=1}^{\infty} \dfrac{1}{1^3 + 2^3 + ... + n^3}$ convergent? Explain.

I.

$\frac{1}{1^3+2^3+\ldots+n^3} < \frac{1}{n^3}$ and $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is convergent (p-series with $p = 3 > 1$)

Therefore $\sum_{n=1}^{\infty} \frac{1}{1^3+2^3+\ldots+n^3}$ is convergent by CT.

II.

Let $a_n = \frac{1}{1^3+2^3+\ldots+n^3}$ and $b_n = \frac{1}{n^3}$. $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$ and $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is convergent (p-series with $p = 3 > 1$). Therefore $\sum_{n=1}^{\infty} \frac{1}{1^3+\ldots+n^3}$ is convergent by LCT.

III.

$$\sum_{n=1}^{\infty} \frac{1}{1^3+2^3+\ldots+n^3} = \sum_{n=1}^{\infty} \frac{1}{\left[\frac{n(n+1)}{2}\right]^2} = \sum_{n=1}^{\infty} \frac{4}{n^4+2n^3+n^2}$$

$\frac{4}{n^4+2n^3+n^2} < \frac{4}{n^4}$ and $\sum \frac{4}{n^4}$ is convergent (p-series with $p = 4 > 1$)

Therefore $\sum_{n=1}^{\infty} \frac{1}{1^3+\ldots+n^3}$ is convergent by CT.

Question 3. Consider the power series $\sum_{n=1}^{\infty} \frac{(-1)^n}{e^n(n+155)}(x-e)^n$.

a) Find the radius of convergence of the series.

$$\lim_{n\to\infty} \left| \frac{(x-e)^{n+1}}{e^{n+1}(n+156)} \cdot \frac{e^n(n+155)}{(x-e)^n} \right| = \lim_{n\to\infty} \left| \frac{x-e}{e} \right| = \frac{|x-e|}{e} < 1$$

$|x-e| < e$ So radius $R = e$.

b) Is this power series conditionally convergent at $x = 2e$? Explain.

$x = 2e$, $\sum_{n=1}^{\infty} \frac{(-1)^n e^n}{e^n(n+155)} = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n+155}$. Let $u_n = \frac{1}{n+155}$.

i) $u_n > 0$ for all $n \in \mathbb{N}$, ii) $u_{n+1} = \frac{1}{n+156} < \frac{1}{n+155} = u_n$ iii) $\lim_{n\to\infty} u_n = 0$.

So by AST the series is convergent.

$\sum_{n=1}^{\infty} \left| (-1)^n \frac{1}{n+155} \right| = \sum_{n=1}^{\infty} \frac{1}{n+155}$. Let $a_n = \frac{1}{n+155}$, $b_n = \frac{1}{n}$.

$\lim_{n\to\infty} \frac{a_n}{b_n} = 1$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent (harmonic series) So the series is divergent.

Therefore the power series is conditionally convergent at $x = 2e$.

Question 4. Evaluate the following integral and limit using power series expansions of the given functions.

a) $\int \ln(1+x^2)dx$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n \Rightarrow \ln(1+x) = \int \frac{1}{1+x} dx = \sum_{n=0}^{\infty} (-1)^n \int x^n dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} + C$$

For $x=0$ we find $c=0$ so $\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$

Putting $x=x^2$ we get $\ln(1+x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{n+1}}{n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{n+1}$

$$\int \ln(1+x^2)dx = \int \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{n+1} dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+3}}{(n+1)(2n+3)} + C$$

b) $\lim_{x\to\infty} x \left(e^{\left(-\frac{1}{x}\right)} - 1 \right)$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow e^{-\frac{1}{x}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{x^n . n!} = 1 - \frac{1}{x.1!} + \frac{1}{x^2 . 2!} - \dots$$

$$\lim_{x\to\infty} x \left(e^{\left(-\frac{1}{x}\right)} - 1 \right) = \lim_{x\to\infty} x \left(\cancel{1} - \frac{1}{x} + \frac{1}{2x^2} - \dots - \cancel{1} \right) = \lim_{x\to\infty} \left(-1 + \frac{1}{2x} - \dots \right) = \underline{-1}$$