Çankaya University
Department of Mathematics
2018 - 2019 Spring Semester

MATH 156 - Calculus for Engineering - II

Second Midterm Examination

1) a) Find the power series expansion of $f(x) = \dfrac{2-3x}{(1-2x)(1-x)}$. What is the radius of convergence of this series?

Hint: $\dfrac{1}{1-u} = \displaystyle\sum_{n=0}^{\infty} u^n, \quad |u| < 1.$

b) Evaluate $\displaystyle\lim_{x\to 0} \frac{x^3e^{-x} + x^4 - x^3}{x^2\sin(x) - \sin\left(x^3\right)}$.

2) a) Let $\vec{u}$, $\vec{v}$ be two vectors such that $\vec{u}\cdot\vec{v} = 1, \quad \vec{u}\times\vec{v} = \vec{i} + \vec{j} + \vec{k} = \langle 1, 1, 1\rangle$.
Find the angle between $\vec{u}$ and $\vec{v}$.

b) The equation of a circle in rectangular coordinates is given by $x^2 - 4\sqrt{3}\,x + y^2 = 0$. Find the equation of the same circle in polar coordinates.

3) a) Find the equation of the plane passing through the points $(0, 0, 35),\ (0, -28, 0),\ (20, 0, 0)$.

b) Find the distance from origin to the line of intersection ℓ of the planes $x + y + z = 1$ and $x - y + z = -1$

4) Evaluate the following limits. (If they exist.)

a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3y^2 + y^6}{(x^2 + y^2)^2}$

b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4y + y^5}{x^4 + y^4}$

5) a) Let $f = f(x, y, z),\ x = u^2 + 4w,\ y = uw,\ z = w^5$. Find f_{ww}.

b) Let $x^2yz^3 - 2xe^y + \ln(xz) = 1 - e^2$. Find the value of z_y at $x = e,\ y = 1,\ z = 1$.

Answers

1) a)

$$\frac{2-3x}{(1-2x)(1-x)} = \frac{A}{1-2x} + \frac{B}{1-x}$$

$$2-3x = A(1-x) + B(1-2x)$$

$$x = 1 \quad \Rightarrow \quad B = 1, \quad x = \frac{1}{2} \quad \Rightarrow \quad A = 1$$

$$\begin{aligned} f(x) = \frac{2-3x}{(1-2x)(1-x)} &= \frac{1}{1-2x} + \frac{1}{1-x} \\ &= \sum_{n=0}^{\infty} \left(2x\right)^n + \sum_{n=0}^{\infty} x^n \\ &= \sum_{n=0}^{\infty} 2^n x^n + \sum_{n=0}^{\infty} x^n \end{aligned}$$

The first series is convergent on $\left|2x\right| < 1 \quad \Rightarrow \quad$ radius of convergence is $\frac{1}{2}$.

The second series is convergent on $\left|x\right| < 1 \quad \Rightarrow \quad$ radius of convergence is 1.

$\Rightarrow \quad$ Radius of convergence of the sum is $\frac{1}{2}$.

1) b) The Taylor series expansions of $\sin x$ and e^x around $x_0 = 0$ are:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots ,$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n \, x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

Therefore:

$$e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n \, x^n}{n!} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots ,$$

$$\sin\left(x^3\right) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(x^3\right)^{2n+1}}{(2n+1)!} = x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \cdots$$

Using these in the limit, we obtain:

$$\begin{aligned}
\lim_{x\to 0} \frac{x^3 e^{-x} + x^4 - x^3}{x^2 \sin(x) - \sin\left(x^3\right)} &= \lim_{x\to 0} \frac{x^3\left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots\right) + x^4 - x^3}{x^2\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) - \left(x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \cdots\right)} \\
&= \lim_{x\to 0} \frac{x^3 - x^4 + \frac{x^5}{2!} - \frac{x^6}{3!} + \cdots + x^4 - x^3}{x^3 - \frac{x^5}{3!} + \frac{x^7}{5!} - \cdots - x^3 + \frac{x^9}{3!} - \frac{x^{15}}{5!} + \cdots} \\
&= \lim_{x\to 0} \frac{\frac{x^5}{2!} - \frac{x^6}{3!} + \cdots}{-\frac{x^5}{3!} + \frac{x^7}{5!} - \cdots} \\
&= \lim_{x\to 0} \frac{\frac{1}{2!} - \frac{x}{3!} + \cdots}{-\frac{1}{3!} + \frac{x^2}{5!} - \cdots} \\
&= -\frac{3!}{2!} = -3
\end{aligned}$$

2) a) $\left|\overrightarrow{u} \times \overrightarrow{v}\right| = \left|\overrightarrow{u}\right|\left|\overrightarrow{v}\right| \sin\theta = \sqrt{1^2+1^2+1^2} = \sqrt{3}$

$$\overrightarrow{u} \cdot \overrightarrow{v} = \left|\overrightarrow{u}\right|\left|\overrightarrow{v}\right| \cos\theta = 1$$

Dividing these equations side by side we obtain:

$$\tan\theta = \sqrt{3} \quad \Rightarrow \quad \theta = \frac{\pi}{3}.$$

b) Using $x = r\cos\theta$ and $y = r\sin\theta$ we obtain:

$$r^2\cos^2\theta - 4\sqrt{3}r\cos\theta + r^2\sin^2\theta = 0$$

$$r^2 = 4\sqrt{3}r\cos\theta \quad \Rightarrow \quad r = 4\sqrt{3}\cos\theta$$

3) a) Let P denote the point $(0, 0, 35)$, Q denote $(0, -28, 0)$ and R denote $(20, 0, 0)$. Then

$$\overrightarrow{PQ} = -28\overrightarrow{j} - 35\overrightarrow{k} = \langle 0,\, -28,\, -35 \rangle$$

$$\overrightarrow{PR} = 2\overrightarrow{i} - 35\overrightarrow{k} = \langle 20,\, 0,\, -35 \rangle$$

The cross product is: $\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ 0 & -28 & -35 \\ 20 & 0 & -35 \end{vmatrix} = 35\left(28\overrightarrow{i} - 20\overrightarrow{j} + 16\overrightarrow{k}\right)$

We can choose the normal vector of the plane as $\overrightarrow{n} = \langle 28,\, -20,\, 16 \rangle$. Then, the equation of the plane is:

$$\langle x-0,\, y-0,\, z-35 \rangle \cdot \langle 28,\, -20,\, 16 \rangle = 0 \quad \Rightarrow \quad 28x - 20y + 16z = 560$$

$$\Rightarrow \quad 7x - 5y + 4z = 140$$

b) The direction vector of the line is: $\overrightarrow{u} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix} = 2\overrightarrow{i} - 2\overrightarrow{k}$

Choosing $x = 0$, we find a point on the line (in other words, on both of the planes) as $(0,\, 1,\, 0)$.
Therefore the equation of the line is: $x = 2t, \quad y = 1, \quad z = -2t$

Now, using the formula $d = \dfrac{\left|\overrightarrow{PQ} \times \overrightarrow{u}\right|}{\left|\overrightarrow{u}\right|}$ with $Q(0,\, 0,\, 0)$ and $P(0,\, 1,\, 0)$ we obtain thedistance as:

$$d = \frac{\sqrt{(-2)^2 + 0^2 + (-2)^2}}{\sqrt{(2)^2 + 0^2 + (-2)^2}} = 1$$

4) a) Using polar coordinates, we obtain

$$\begin{aligned}
\lim_{r\to 0} \frac{r^3\cos^3\theta\cdot r^2\sin^2\theta + r^6\sin^6\theta}{\left(r^2\right)^2} &= \lim_{r\to 0} \frac{r^5\Big(\cos^3\theta\sin^2\theta + r\sin^6\theta\Big)}{r^4} \\
&= \lim_{r\to 0} r\Big(\cos^3\theta\sin^2\theta + r\sin^6\theta\Big) \\
&= 0
\end{aligned}$$

b) We can easily see that $\dfrac{x^4}{x^4+y^4} \leqslant 1$ and $\dfrac{y^4}{x^4+y^4} \leqslant 1$ therefore:

$$0 \leqslant \left|\frac{x^4y+y^5}{x^4+y^4}\right| \leqslant \left|\frac{x^4}{x^4+y^4}\right||y| + \left|\frac{y^4}{x^4+y^4}\right||y| \leqslant |y|+|y| \leqslant 2|y|$$

Using $\lim\limits_{(x,y)\to(0,0)} |y| = 0$ and then Sandwich theorem, we obtain:

$$\lim_{(x,y)\to(0,0)} \frac{x^4y+y^5}{x^4+y^4} = 0$$

5) a) $f_w = f_x\,x_w + f_y\,y_w + f_z\,z_w = 4f_x + uf_y + 5w^4 f_z$

$$\begin{aligned}
f_{ww} &= 4\big(f_{xx}\,x_w + f_{xy}\,y_w + f_{xz}\,z_w\big) + u\big(f_{yx}\,x_w + f_{yy}\,y_w + f_{yz}\,z_w\big) \\
&\quad +20w^3 f_z + 5w^4\big(f_{zx}\,x_w + f_{zy}\,y_w + f_{zz}\,z_w\big) \\
&= 16f_{xx} + 4uf_{xy} + 20w^4 f_{xz} + 4uf_{yx} + u^2 f_{yy} + 5uw^4 f_{yz} \\
&\quad +20w^3 f_z + 20w^4 f_{zx} + 5uw^4 f_{zy} + 25w^8 f_{zz}
\end{aligned}$$

b) Let $F(x,y,z) = x^2yz^3 - 2xe^y + \ln(xz)$. Then

$$z_y = -\frac{F_y}{F_z} = -\frac{x^2z^3 - 2xe^y}{3x^2yz^2 + \dfrac{1}{z}}$$

At the point $x=e,\ y=1,\ z=1$ we obtain:

$$z_y\Big|_{(x,y,z)=(e,1,1)} = -\frac{e^2-2e^2}{3e^2+1} = \frac{e^2}{3e^2+1}$$