Çankaya University
Department of Mathematics
2018 - 2019 Spring Semester

MATH 156 - Calculus for Engineering - II

First Midterm Examination

1) Are the following sequences convergent or divergent? Explain.

a) $a_n = \dfrac{n^2+1}{n^2+4}$

b) $b_n = \dfrac{5^n}{n!}$

2) The following series are convergent. Find their sum:

a) $\displaystyle\sum_{n=1}^{\infty} \frac{3^{n-1}}{4^{n+1}}$

b) $\displaystyle\sum_{n=0}^{\infty} \frac{2}{n^2+6n+5}$

3) Are the following series convergent or divergent? Explain. Indicate which tests you use.

a) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n - 5}{\ln n + 4}$

b) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{5^n + n^5}$

c) $\displaystyle\sum_{n=1}^{\infty} \frac{4\sqrt{n} - \ln n}{7n\sqrt{n} + \sqrt[3]{n}}$

4) a) Is the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n\, n^4}{e^n}$ absolutely convergent, conditionally convergent or divergent? Explain.

b) Is the series $\displaystyle\sum_{n=0}^{\infty} \Big[\ln\big(3n+5\big) - \ln\big(n+1\big)\Big]^n$ convergent or divergent? Explain.

c) Is the series $\displaystyle\sum_{n=1}^{\infty} \frac{(n+3)!\,(n-1)!}{(2n)!\,n^2}$ convergent or divergent? Explain.

5) Find the radius and interval of convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{n(2x-5)^n\, 2^{2n}}{3^n}$.

Answers

1)

a) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \dfrac{n^2+1}{n^2+4} = \lim_{n\to\infty} \dfrac{n^2\left(1+\dfrac{1}{n^2}\right)}{n^2\left(1+\dfrac{4}{n^2}\right)} = \lim_{n\to\infty} \dfrac{1+\dfrac{1}{n^2}}{1+\dfrac{4}{n^2}} = \dfrac{1}{1} = 1.$

The sequence is convergent.

b)
$$\begin{aligned}\lim_{n\to\infty} b_n &= \lim_{n\to\infty} \frac{5\cdot 5\cdots 5\cdot 5}{n\cdot(n-1)\cdots 2\cdot 1} \\ &= \lim_{n\to\infty} \frac{5}{n}\cdot\frac{5}{n-1}\cdots\frac{5}{2}\cdot\frac{5}{1} \\ &= 0\end{aligned}$$

The sequence is convergent.

OR

$\dfrac{b_{n+1}}{b_n} = \dfrac{5}{n+1} < 1$ for $n \geqslant 4$ so the sequence is decreasing monotonically.

$\dfrac{5^n}{n!} > 0$ so the sequence is bounded below. Therefore it is convergent by monotonic sequence theorem.

OR

Consider $c_n = \dfrac{5^n}{n\cdot 5\cdot 5\cdots 5\cdot 4\cdot 3\cdot 2\cdot 1} = \dfrac{5^n}{n\cdot 5^{n-5}\cdot 4\cdot 3\cdot 2\cdot 1} = \dfrac{5^5}{24n}.$

Clearly, $0 < b_n < c_n$ and $\lim_{n\to\infty} c_n = 0$. So the sequence b_n is convergent to 0.

2)

a)
$$\sum_{n=1}^{\infty} \frac{3^{n-1}}{4^{n+1}} = \frac{1}{12}\sum_{n=1}^{\infty}\left(\frac{3}{4}\right)^n$$
$$= \frac{1}{12}\left[\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n - 1\right]$$
$$= \frac{1}{12}\left[\frac{1}{1-\frac{3}{4}} - 1\right]$$
$$= \frac{1}{12}\cdot 3$$
$$= \frac{1}{4}$$

b)
$$\frac{2}{n^2+6n+5} = \frac{A}{n+1} + \frac{B}{n+5} \quad \Rightarrow \quad 2 = A(n+5) + B(n+1)$$

The solution of these equations give $A = \frac{1}{2},\ B = -\frac{1}{2}$ in other words the series is: $\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{n+1} - \frac{1}{n+5}\right)$.

Let's find the n^{th} partial sum S_n:

$$S_n = \frac{1}{2}\left[\left(\frac{1}{1}-\frac{1}{5}\right) + \left(\frac{1}{2}-\frac{1}{6}\right) + \left(\frac{1}{3}-\frac{1}{7}\right) + \left(\frac{1}{4}-\frac{1}{8}\right) + \left(\frac{1}{5}-\frac{1}{9}\right) + \cdots + \left(\frac{1}{n+1}-\frac{1}{n+5}\right)\right]$$
$$S_n = \frac{1}{2}\left[1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{n+2} - \frac{1}{n+3} - \frac{1}{n+4} - \frac{1}{n+5}\right]$$
$$\lim_{n\to\infty} S_n = \frac{1}{2}\left[1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right] = \frac{25}{24}$$

3)

a) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\ln n - 5}{\ln n + 4} = 1.$

The series is divergent by n^{th} term test.

b) The series $\sum_{n=1}^{\infty} \left(\frac{2}{5}\right)^n$ is convergent by geometric series test.

$$5^n + n^5 > 5^n \quad \Rightarrow \quad \frac{2^n}{5^n + n^5} < \frac{2^n}{5^n}$$

Therefore $\sum_{n=1}^{\infty} \frac{2^n}{5^n + n^5}$ is also convergent by comparison test.

c) The series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent by integral test. (Harmonic series)

$$\lim_{n\to\infty} \frac{\dfrac{4\sqrt{n} - \ln n}{7n\sqrt{n} + \sqrt[3]{n}}}{\dfrac{1}{n}} = \lim_{n\to\infty} \frac{4n\sqrt{n} - n\ln n}{7n\sqrt{n} + \sqrt[3]{n}} = \frac{4}{7}$$

Therefore the given series is also divergent by limit comparison test.

4)

a) Consider the series $\sum_{n=0}^{\infty}\left|\frac{(-1)^n\,n^4}{e^n}\right| = \sum_{n=0}^{\infty}\frac{n^4}{e^n}$. Let's use ratio test.

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{\dfrac{(n+1)^4}{e^{n+1}}}{\dfrac{n^4}{e^n}} = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^4\frac{1}{e} = \frac{1}{e}$$

$e > 2 \quad \Rightarrow \quad \frac{1}{e} < 1$. Therefore this series is convergent by ratio test. So the given series is absolutely convergent.

b) Let's use root test.

$$\lim_{n\to\infty}\left(a_n\right)^{\frac{1}{n}} = \lim_{n\to\infty}\ln\left(3n+5\right) - \ln\left(n+1\right) = \lim_{n\to\infty}\ln\left(\frac{3n+5}{n+1}\right) = \ln 3$$

$e < 3 \quad \Rightarrow \quad \ln 3 > 1$. Therefore this series is divergent by root test.

c) Let's use ratio test.

$$\begin{aligned}\lim_{n\to\infty}\frac{a_{n+1}}{a_n} &= \lim_{n\to\infty}\frac{\dfrac{(n+4)!\,n!}{(2n+2)!\,(n+1)^2}}{\dfrac{(n+3)!\,(n-1)!}{(2n)!\,n^2}}\\ &= \lim_{n\to\infty}\frac{(n+4)!}{(n+3)!}\,\frac{n!}{(n-1)!}\,\frac{(2n)!}{(2n+2)!}\,\frac{n^2}{(n+1)^2}\\ &= \lim_{n\to\infty}\frac{(n+4)\,n}{(2n+2)(2n+1)}\left(\frac{n}{n+1}\right)^2\\ &= \frac{1}{4}\end{aligned}$$

$\frac{1}{4} < 1$. Therefore this series is convergent by ratio test.

5) Using root test, we obtain: $\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \dfrac{\sqrt[n]{n}\,|2x-5|\,4}{3} = \dfrac{4}{3}\,|2x-5| < 1.$

So the series converges for $|2x-5| < \dfrac{3}{4}$ by root test.

$$|2x-5| < \frac{3}{4} \quad \Rightarrow \quad -\frac{3}{4} < 2x-5 < \frac{3}{4} \quad \Rightarrow \quad \frac{17}{8} < x < \frac{23}{8}$$

Now we have to check endpoints:

$$x = \frac{23}{8} \quad \Rightarrow \quad \sum_{n=1}^{\infty} n \left(\frac{3}{4}\right)^n \frac{2^{2n}}{3^n} = \sum_{n=1}^{\infty} n.$$

$$x = \frac{17}{8} \quad \Rightarrow \quad \sum_{n=1}^{\infty} n \left(-\frac{3}{4}\right)^n \frac{2^{2n}}{3^n} = \sum_{n=1}^{\infty} (-1)^n\, n.$$

Both of these series are obviously divergent by n^{th} term test.

Interval of convergence is: $\left(\dfrac{17}{8}, \dfrac{23}{8}\right)$.

Radius of convergence is: $R = \dfrac{\frac{23}{8} - \frac{17}{8}}{2} = \dfrac{3}{8}.$