Çankaya University
Department of Mathematics
2018 - 2019 Spring Semester

MATH 156 - Calculus for Engineering - II

Final Examination

1) a) Find the equation of the tangent plane and the normal line to the surface $z = \dfrac{4}{\pi} \arctan\left(xy\right)$ at the point $\left(1,\, 1,\, 1\right)$.

b) Evaluate the integral $\displaystyle\int_0^1 e^{-2x^2} dx$ in the form of a series. (Hint: You may use Maclaurin series.)

2) a) Find the absolute maximum and minimum values of the function $f(x, y) = e^{x^3+y^3}$ in the disk $x^2 + y^2 \leqslant 4$.

b) Find the maximum and minimum values of the function $f(x, y, z) = x^2 + y^2 + 4z$ subject to the constraint $5x + 6y + z = 156$ if they exist.

3) Find and classify the critical points of the function $f(x, y) = 25x^3 + 6xy + 3xy^2$.

4) a) Evaluate the integral $\displaystyle\int_0^{2\pi} \int_{x/2}^{\pi} \frac{\sin y}{y}\, dy\, dx.$

b) Set up the double integral that gives the area of the region inside the circle $(x-1)^2 + y^2 = 1$ and above the line $y = \sqrt{3}\,x$. Do NOT evaluate the integral.

5) a) Find the volume of the solid that is below the cone $z = 3 - \sqrt{x^2+y^2}$, above the plane $z = 0$ and inside the cylinder $x^2 + y^2 = 4$.

b) Evaluate $\displaystyle\iint_R \left(2x+y\right) dy\, dx$ where R is the region bounded by the lines:

$2x + y = 5, \quad 2x + y = 10, \quad 2y - x = 0$ and $2y - x = 20.$

Answers

1) a) $F(x,y,z) = \frac{4}{\pi}\arctan\big(xy\big) - z = 0$

$$\nabla F = \frac{4}{\pi}\frac{y}{1+(xy)^2}\,\vec{i} + \frac{4}{\pi}\frac{x}{1+(xy)^2}\,\vec{j} - \vec{k}$$

Normal vector: $\quad \vec{n} = \nabla F\big(1,\,1,\,1\big) = \frac{2}{\pi}\,\vec{i} + \frac{2}{\pi}\,\vec{j} - \vec{k}$

Tangent plane: $\frac{2}{\pi}(x-1) + \frac{2}{\pi}(y-1) - (z-1) = 0$

$$\Rightarrow \quad x + y - \frac{\pi}{2}z = 2 - \frac{\pi}{2}$$

Normal line: $\begin{aligned} x &= 1 + \frac{2}{\pi}\,t \\ y &= 1 + \frac{2}{\pi}\,t \\ z &= 1 - t \end{aligned}$

b) Maclaurin series of the exponential function is: $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad x \in \mathbb{R}$ therefore:

$$e^{-2x^2} = \sum_{n=0}^{\infty} \frac{\big(-2x^2\big)^n}{n!} = \sum_{n=0}^{\infty} \frac{\big(-2\big)^n}{n!}\,x^{2n}$$

$$\begin{aligned} \Rightarrow \quad \int_0^1 e^{-2x^2}\,dx &= \int_0^1 \sum_{n=0}^{\infty} \frac{\big(-2\big)^n}{n!}\,x^{2n} \\ &= \sum_{n=0}^{\infty} \frac{\big(-2\big)^n}{n!}\,\frac{x^{2n+1}}{2n+1}\bigg|_0^1 \\ &= \sum_{n=0}^{\infty} \frac{(-2)^n}{\big(2n+1\big)\,n!} \end{aligned}$$

2) a) $f_x = 3x^2\, e^{x^3+y^3} = 0 \quad \Rightarrow \quad x = 0, \quad f_y = 3y^2\, e^{x^3+y^3} = 0 \quad \Rightarrow \quad y = 0.$

The only critical point of f is $(0, 0)$.

On the boundary, $x = 2\cos\theta, \quad y = 2\sin\theta$, therefore: $f(2\cos\theta,\, 2\sin\theta) = g(\theta) = e^{8(\cos^3\theta+\sin^3\theta)}$

$$\Rightarrow \quad g'(\theta) = e^{8(\cos^3\theta+\sin^3\theta)}\, 8\Big[3\cos^2\theta(-\sin\theta) + 3\sin^2\theta\cos\theta\Big] = 0$$

$$\cos\theta\,\sin\theta\Big[\cos\theta - \sin\theta\Big] = 0$$

$\Rightarrow \quad \sin\theta = 0$ or $\cos\theta = 0$ or $\sin\theta = \cos\theta$.

$$\Rightarrow \quad \theta = 0, \quad \theta = \pi, \quad \theta = \frac{\pi}{2}, \quad \theta = \frac{3\pi}{2}, \quad \theta = \frac{\pi}{4}, \quad \theta = \frac{5\pi}{4}.$$

Let's make a table and compute $f(x, y)$ at those points:

θ	$(x,\, y)$	$f(x,\, y)$
	$(0,\, 0)$	1
0	$(2,\, 0)$	e^8
π	$(-2,\, 0)$	e^{-8}
$\frac{\pi}{2}$	$(0,\, 2)$	e^8
$\frac{3\pi}{2}$	$(0,\, -2)$	e^{-8}
$\frac{\pi}{4}$	$(\sqrt{2},\, \sqrt{2})$	$e^{4\sqrt{2}}$
$\frac{5\pi}{4}$	$(-\sqrt{2},\, -\sqrt{2})$	$e^{-4\sqrt{2}}$

Maximum is: $f(2, 0) = f(0, 2) = e^8$, Minimum is: $f(-2, 0) = f(0, -2) = e^{-8}$

b) $\nabla f = \lambda \nabla g$. Using Lagrange multipliers, we obtain 4 equations in 4 unknowns:

$$\begin{aligned} 2x &= 5\lambda \\ 2y &= 6\lambda \\ 4 &= \lambda \\ 5x + 6y + z &= 156 \end{aligned}$$

The only solution is: $x = 10, \quad y = 12, \quad z = 34$. Minimum is $f(10, 12, 34) = 380$.

There is no maximum value. z, therefore $f(x, y, z)$ can take arbitrarily large values.

3)

$$\begin{aligned} f_x &= 75x^2 + 6y + 3y^2 &= 0 \\ f_y &= 6x + 6xy &= 0 \end{aligned}$$

Second equation gives:

$$6x(1+y) = 0 \quad \Rightarrow \quad x = 0 \quad \text{or} \quad y = -1.$$

We will insert these in the first equation:

$$x = 0 \quad \Rightarrow \quad 6y + 3y^2 = 0 \quad \Rightarrow \quad y = 0 \quad \text{or} \quad y = -2.$$

$$y = -1 \quad \Rightarrow \quad 75x^2 - 6 + 3 = 0 \quad \Rightarrow \quad x^2 = \frac{1}{25} \quad \text{or} \quad x = \pm\frac{1}{5}.$$

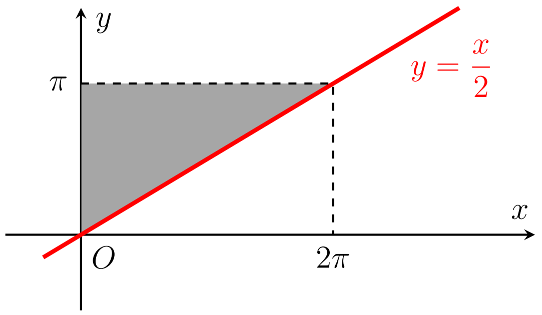
Critical points are: $(0,\, 0),\ (0,\, -2),\ \left(\frac{1}{5},\, -1\right)$ and $\left(-\frac{1}{5},\, -1\right)$.

$$\begin{aligned} f_{xx} &= 150x \\ f_{yy} &= 6x \\ f_{xy} &= 6 + 6y \end{aligned}$$

Now we have to check the critical points separately:

$(0,\, 0)$	$(0,\, -2)$	$\left(\frac{1}{5},\, -1\right)$	$\left(-\frac{1}{5},\, -1\right)$
$A = 0$	$A = 0$	$A = 30$	$A = -30$
$B = 0$	$B = 0$	$B = \frac{6}{5}$	$B = -\frac{6}{5}$
$C = 6$	$C = -6$	$C = 0$	$C = 0$
$\Delta = -36$	$\Delta = -36$	$\Delta = 36$	$\Delta = 36$
$\Delta < 0$	$\Delta < 0$	$\Delta > 0,\ A > 0$	$\Delta > 0,\ A < 0$
$\Rightarrow$ Saddle Point	$\Rightarrow$ Saddle Point	$\Rightarrow$ Local Minimum.	$\Rightarrow$ Local Maximum.

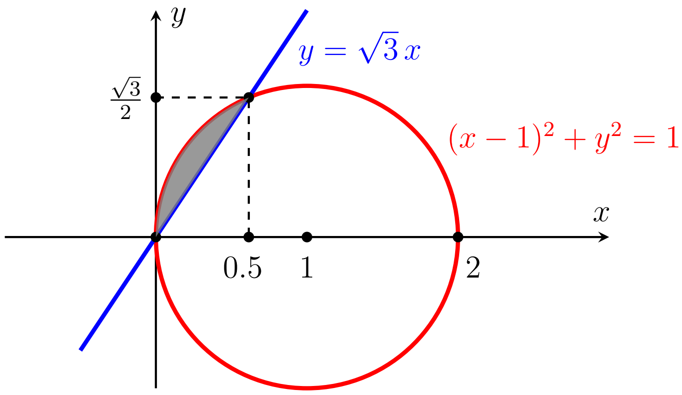
4) a)



$$0 \leqslant x \leqslant 2\pi, \quad \frac{x}{2} \leqslant y \leqslant \pi$$

$$\begin{aligned}
\int_0^{2\pi}\int_{x/2}^{\pi} \frac{\sin y}{y}\,dy\,dx &= \int_0^{\pi}\int_0^{2y} \frac{\sin y}{y}\,dx\,dy \\
&= \int_0^{\pi} \frac{\sin y}{y}\,2y\,dy \\
&= -2\cos y \Big|_0^{\pi} \\
&= -2\cos\pi + 2\cos 0 \\
&= 4
\end{aligned}$$

b)



$y = \sqrt{3}\,x \quad \Rightarrow \quad (x-1)^2 + \left(\sqrt{3}\,x\right)^2 = 1$

$x^2 - 2x + 1 + 3x^2 = 1$

$4x^2 - 2x = 0 \quad \Rightarrow \quad 2x\left(2x - 1\right) = 0$

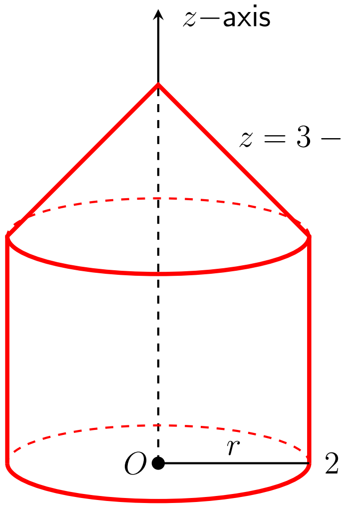
$x = 0$ or $x = \frac{1}{2}.$

The area is: $A = \int_0^{1/2}\int_{\sqrt{3}\,x}^{\sqrt{1-(x-1)^2}} dy\,dx$

We can express the same area by using polar coordinates:

$$A = \int_{\pi/3}^{\pi/2}\int_0^{2\cos\theta} r dr\,d\theta$$

5) a)



We will use polar coordinates to express the volume:

$x = r\cos\theta, \quad y = r\sin\theta,$

$0 \leqslant r \leqslant 2, \quad 0 \leqslant \theta \leqslant 2\pi$

$$\begin{aligned}
\iint_D \left(3 - \sqrt{x^2+y^2}\right) dA &= \int_0^{2\pi}\int_0^2 (3-r) r dr\, d\theta \\
&= \int_0^{2\pi} \left(\frac{3}{2}r^2 - \frac{r^3}{3}\right)\Big|_0^2 d\theta \\
&= \int_0^{2\pi} \left(\frac{12}{2} - \frac{8}{3}\right) d\theta \\
&= 2\pi\left(\frac{20}{6}\right) \\
&= \frac{20\pi}{3}
\end{aligned}$$

b) Lets use new variables $u = 2x + y, \quad v = 2y - x$. Clearly, $\quad 5 \leqslant u \leqslant 10, \quad 0 \leqslant v \leqslant 20.$

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}} = \frac{1}{\begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix}} = \frac{1}{5}$$

$$\begin{aligned}
\iint_R (2x+y) dydx &= \int_5^{10}\int_0^{20} \frac{u}{5}\, dv\, du \\
&= \frac{1}{5}\int_5^{10} (20-0) u\, du \\
&= 4 \cdot \frac{u^2}{2}\Big|_5^{10} \\
&= 2(100-25) \\
&= 150
\end{aligned}$$