Çankaya University
Department of Mathematics
2016 - 2017 Summer Semester

MATH 156 - Calculus for Engineering II

Second Midterm Examination

1) a) Find the equation of the line passing through the points $P_0(0, 5, 10)$ and $P_1(-4, 2, 8)$

Answer: We can choose the parallel vector to the line as $\overrightarrow{v} = \overrightarrow{P_0P_1} = (-4, -3, -2)$.

Then $\quad \overrightarrow{P_0P} = t\overrightarrow{v} \Longrightarrow x = -4t, \quad y = 5 - 3t, \quad z = 10 - 2t.$

(6 pts.)**b)** Find the angle between the lines $l_1 : x = 3 - 2t, \quad y = 1 + 2t, \quad z = 4t$ and $l_2 : x = 3, \quad y = 1 + s, \quad z = 7s.$

Answer: The angle between the lines equal to the angle between parallel vectors $\overrightarrow{v_1} = (-2, 2, 4)$ and $\overrightarrow{v_2} = (0, 1, 7)$.

To find the angle between the vectors we can use: $\cos(\theta) = \dfrac{\overrightarrow{v_1} \cdot \overrightarrow{v_2}}{|\overrightarrow{v_1}|\,|\overrightarrow{v_2}|} = \dfrac{\sqrt{3}}{2} \quad \Rightarrow \quad \theta = \dfrac{\pi}{6}.$

(7 pts.) **c)** Find the equation of the plane passing through the points $P_0(1, 1, -2), \quad P_1(3, -3, 0)$ and $P_2(1, 3, -1)$.

Answer: The normal vector $\overrightarrow{n} = \overrightarrow{P_0P_1} \times \overrightarrow{P_0P_1} = -8\overrightarrow{i} - 2\overrightarrow{j} + 4\overrightarrow{k}$.

Then equation of the plane: $P_0P \cdot \overrightarrow{n} = 0 \quad \Rightarrow \quad 4x + y - 2z = 9$

2) Evaluate the following limits: (if they exist)

(9 pts.) **a)** $\displaystyle\lim_{(x,y)\to(0,0)} \frac{10x^2 - 4y^2}{3x^2 + 5y^2}$

Answer: Along the paths $y = mx$,

$$\lim_{(x,y)\to(0,0)} \frac{10x^2 - 4y^2}{3x^2 + 5y^2} = \lim_{x\to 0} \frac{10x^2 - 4m^2x^2}{3x^2 + 5m^2x^2} = \frac{10 - 4m^2}{3 + 5m^2}$$

Then when m changes limit changes and limit does not exist.

(9 pts.) **b)** $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^5 + 7x^2y^3}{(x^2 + y^2)^2}$

Answer: By polar coordinates:

$$\lim_{(x,y)\to(0,0)} \frac{x^5 + 7x^2y^3}{(x^2 + y^2)^2} = \lim_{r\to 0} \frac{r^5\cos^5(\theta) + 7r^2\cos^2(\theta)r^3\sin^3(\theta)}{r^4} = 0$$

3) (8 pts.) **a)** Let $f(x,y) = x^2y^4\tan\left(x^3y^6\right)$. Find f_x and f_y.

Answer:
$$f_x = 2xy^4\tan\left(x^3y^6\right) + 3x^4y^{10}\sec^2\left(x^3y^6\right)$$
$$f_y = 4x^2y^3\tan\left(x^3y^6\right) + 5x^5y^9\sec^2\left(x^3y^6\right)$$

(8 pts.) **b)** Let $g(x,y) = x^2y + ye^x, \quad x = t^2\ln r, \quad y = r^3 + r$. Find g_t and g_r.

Answer:
$$g_t = (2xy + ye^x)\,2t\ln r$$
$$g_r = \left(2xy + ye^x\right)\frac{t^2}{r} + (x^2 + e^x)(3r^2 + 1)$$

where $\quad x = t^2\ln r, \quad y = r^3 + r.$

4) (8 pts.) **a)** If $\pi\sin(x+2y) + x\ln(x^2+z^2) + \dfrac{8y}{1+z^2} = 2\pi$, find $\dfrac{\partial z}{\partial y}$ at the point $\left(0, \dfrac{\pi}{4}, 1\right)$.

Answer: Let $F(x,y,z) = \pi\sin(x+2y) + x\ln(x^2+z^2) + \dfrac{8y}{1+z^2} - 2\pi = 0$. Then

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{2\pi\cos(x+2y) + \dfrac{8}{1+z^2}}{\dfrac{2xz}{x^2+z^2} - \dfrac{16yz}{\left(1+z^2\right)^2}}$$ and at the point $\left(0, \dfrac{\pi}{4}, 1\right)$, $\quad \dfrac{\partial z}{\partial y} = \dfrac{4}{\pi}$.

(8 pts.) **b)** Find a unit vector in the direction where $f(x,y,z) = x^2 + 2xy^2 + 3z^2 + yz$ increases fastest at the point $P_0(2,2,1)$.

Answer: $f(x,y,z)$ increases fastest in the direction of $\quad \overrightarrow{\nabla f}(P_0) = (12, 17, 8)$.

Unit vector in this direction $\overrightarrow{u} = \dfrac{1}{\sqrt{497}}(12, 17, 8)$

5) Find and classify the critical points of $f(x,y) = 16x^3 + 16x^2 - 4xy + y^2$ using the second derivative test.

Answer: $f_x = 0 \Rightarrow 48x^2 + 32x - 4y = 0$
$f_y = 0 \Rightarrow -4x + 2y = 0$

The second equation gives $y = 2x$. Then by first equation $x = 0$ and $x = -1/2$.

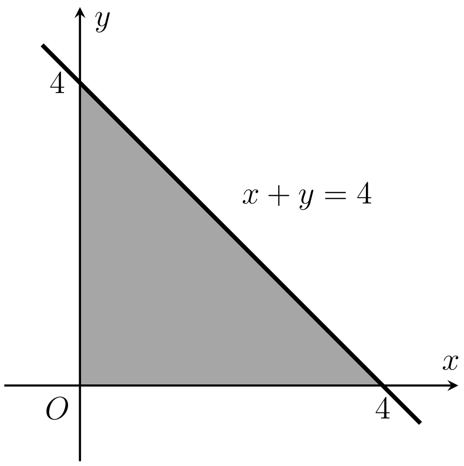
Critical points: $P_1(0,0)$, $P_2(\frac{-1}{2}, -1)$

$f_{xx} = 96x + 32, \quad f_{xy} = -4, \quad f_{yy} = 2 \quad$ and $\quad D(x,y) = 2(96x+32) - 16$

$D(P_1) = 48 > 0,\ f_{xx}(P_1) = 32 > 0 \Rightarrow$ local minimum.

$D(P_2) = -48 < 0 \Rightarrow$ saddle point.

6) Find the absolute extrema of the function $f(x, y) = (x-1)^2 + (y-2)^2$ on the triangular region surrounded by $x-$axis, $y-$axis and the line $x + y = 4$.



Answer: $f_x = 0 \;\Rightarrow\; 2(x-1) = 0 \;\Rightarrow x = 1$
$f_y = 0 \;\Rightarrow\; 2(y-2) = 0 \;\Rightarrow y = 2$

The only critical point is $(1, 2)$ and it is inside the region.
Now let's look at the boundary:

- $x = 0, \quad 0 \le y \le 4$

  $g_1(y) = y^2 - 4y + 5 \Rightarrow g_1' = 2y - 4 = 0$

  $(0, 2)$ is on the boundary.

- $y = 0, \quad 0 \le x \le 4$

  $g_2(x) = x^2 - 2x + 5 \Rightarrow g_2' = 2x - 2 = 0$

  $(1, 0)$ is on the boundary.

- $y = 4 - x, \quad 0 \le x \le 4$

  $g_3(x) = 2x^2 - 6x + 5 \Rightarrow g_3' = 4x - 6 = 0$

  $\left(\frac{3}{2}, \frac{5}{2}\right)$ is on the boundary.

Also considering the endpoints $(0, 0)$, $(4, 0)$ and $(0, 4)$, we obtain:

Maximum: $f(4, 0) = 13$

Minimum: $f(1, 2) = 0$