Çankaya University
Department of Mathematics
2016 - 2017 Spring Semester

MATH 156 - Calculus for Engineering II

Second Midterm Examination

1) a) Find the equation of the line passing through the points $P(1,-3,6)$ and $Q(2,10,-4)$.

b) Find the angle between the planes $\sqrt{3}\,x+\sqrt{3}\,y+\sqrt{2}\,z=5$ and $3\,x+3\,y-\sqrt{6}\,z=10$.

c) Find the distance between the point $S(1,0,2)$ and the plane that passes through the points $P(0,0,1)$, $Q(0,1,0)$ and $R(1,0,0)$.

2) Evaluate the following limits: (if they exist)

a) $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^4+x^2y}{x^2+y^4}$

b) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3y}{x^4+y^4}$

3) a) Let $f(x,y)=\ln|\sin x+x^2y|$. Find f_x and f_y.

b) Let $g=g(u,v),\quad u=x+2y,\quad v=3x-y$. Find g_{xx} and g_{yy}.

4) a) If $x\cos(yz)+ye^{xy}+x^3z=4$, find $\dfrac{\partial z}{\partial x}$ at the point $(1,0,3)$.

b) Find the directional derivative of the function $f(x,y,z)=x^2y+3y^2z$ at the point $(4,1,0)$ in the direction of the vector $\overrightarrow{v}=2\overrightarrow{i}+\overrightarrow{j}+\overrightarrow{k}$.

5) Find and classify the critical points of $f(x,y)=12x+\dfrac{x^3}{12}+16xy+4xy^2$ using the second derivative test.

6) Find the absolute extrema of the function $f(x,y)=e^{x^3-y^3}$ on the disk $x^2+y^2\leqslant 1$.

Answers

1) a) $\overrightarrow{v} = \overrightarrow{PQ} = \overrightarrow{i} + 13\overrightarrow{j} - 10\overrightarrow{k}$

$$\Rightarrow \quad \begin{aligned} x &= 1 + t \\ y &= -3 + 13t \\ z &= 6 - 10t \end{aligned}$$

b) Normal of the first plane: $\overrightarrow{u} = \sqrt{3}\,\overrightarrow{i} + \sqrt{3}\,\overrightarrow{j} + \sqrt{2}\,\overrightarrow{k}$

Normal of the second plane: $\overrightarrow{v} = 3\overrightarrow{i} + 3\overrightarrow{j} - \sqrt{6}\,\overrightarrow{k}$

$$\cos\theta = \frac{\overrightarrow{u}\cdot\overrightarrow{v}}{|\overrightarrow{u}|\,|\overrightarrow{v}|} = \frac{3\sqrt{3} + 3\sqrt{3} - \sqrt{12}}{\sqrt{8}\,\sqrt{24}} = \frac{1}{2}$$

$$\Rightarrow \quad \theta = \frac{\pi}{3}$$

b) $\overrightarrow{PQ} = \overrightarrow{j} - \overrightarrow{k}, \quad \overrightarrow{PR} = \overrightarrow{i} - \overrightarrow{k}$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = -\overrightarrow{i} - \overrightarrow{j} - \overrightarrow{k}$$

The plane passing through the points P, Q and R is:

$$-x - y - (z - 1) = 0 \quad \Rightarrow \quad x + y + z = 1$$

$\overrightarrow{PS} = \overrightarrow{i} + \overrightarrow{k} \quad \Rightarrow$ the distance between S and the plane is:

$$d = \frac{|(\overrightarrow{i} + \overrightarrow{k})\cdot(\overrightarrow{i} + \overrightarrow{j} + \overrightarrow{k})|}{|\overrightarrow{i} + \overrightarrow{j} + \overrightarrow{k}|} = \frac{2}{\sqrt{3}}$$

2) a) $y^4 \leqslant x^2 + y^4 \quad \Rightarrow \quad \dfrac{y^4}{x^2 + y^4} \leqslant 1$

$$x^2 \leqslant x^2 + y^4 \quad \Rightarrow \quad \frac{x^2}{x^2 + y^4} \leqslant 1$$

$$\frac{xy^4 + x^2y}{x^2 + y^4} = x \cdot \frac{y^4}{x^2 + y^4} + y \cdot \frac{x^2}{x^2 + y^4} \leqslant |x| + |y|$$

$$\Rightarrow \quad 0 \leqslant \left|\frac{xy^4 + x^2y}{x^2 + y^4}\right| \leqslant |x| + |y|$$

$\Rightarrow \quad \displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^4 + x^2y}{x^2 + y^4} = 0$ by sandwich theorem.

b) Let's approach origin along the line $y = x$: $\displaystyle\lim_{x\to 0} \frac{x^4}{x^4 + x^4} = \frac{1}{2}$

Let's approach origin along the line $y = -x$: $\displaystyle\lim_{x\to 0} \frac{-x^4}{x^4 + x^4} = -\frac{1}{2}$

We obtain two different results therefore the limit does not exist.

3) a) $f_x = \dfrac{\cos x + 2xy}{\sin x + x^2 y}$

$f_y = \dfrac{x^2}{\sin x + x^2 y}$

b) $g_x = g_u u_x + g_v v_x = g_u + 3g_v$

$g_{xx} = g_{uu} + 3g_{uv} + 3g_{vu} + 9g_{vv}$

$g_y = g_u u_y + g_v v_y = 2g_u - g_v$

$g_{yy} = 4g_{uu} - 2g_{uv} - 2g_{vu} + g_{vv}$

4) a) $\dfrac{\partial z}{\partial x} = -\dfrac{\cos(yz) + y^2 e^{xy} + 3x^2 z}{-xy\sin(yz) + x^3}$

$\left.\dfrac{\partial z}{\partial x}\right|_{(1,0,3)} = -\dfrac{1+0+9}{0+1} = -10$

b) $\nabla f = 2xy\,\overrightarrow{i} + (x^2 + 6yz)\,\overrightarrow{j} + 3y^2\,\overrightarrow{k}$

$\nabla f(4,1,0) = 8\,\overrightarrow{i} + 16\,\overrightarrow{j} + 3\,\overrightarrow{k}$

$D_u f = \nabla f(4,1,0) \cdot \overrightarrow{u} = (8\,\overrightarrow{i} + 16\,\overrightarrow{j} + 3\,\overrightarrow{k}) \cdot \dfrac{2\,\overrightarrow{i} + \overrightarrow{j} + \overrightarrow{k}}{\sqrt{6}} = \dfrac{35}{\sqrt{6}}$

5) $f_x = 12 + \dfrac{x^2}{4} + 16y + 4y^2 = 0$

$f_y = 16x + 8xy = 0 \quad \Rightarrow \quad x = 0$ or $y = -2$

$x = 0 \quad \Rightarrow \quad y = -1$ or $y = -3$

$y = -2 \quad \Rightarrow \quad x = \pm 4$

The critical points are: $(0,-1),\ (0,-3),\ (4,-2),\ (-4,-2)$

$f_{xx} = \dfrac{x}{2}, \quad f_{yy} = 8x, \quad f_{xy} = 16 + 8y$

Using second derivative test, we obtain:

$f(0,-1) = 0$: saddle point
$f(0,-3) = 0$: saddle point
$f(4,-2) = -\dfrac{32}{3}$: local minimum
$f(-4,-2) = \dfrac{32}{3}$: local maximum

6) $f_x = 0 \quad \Rightarrow \quad x = 0, \quad f_y = 0 \quad \Rightarrow \quad y = 0$

The only critical point of f is $(0,0)$.

On the boundary, Let's define a new function:
$g(\theta) = f(\cos\theta, \sin\theta) = e^{\cos^3\theta - \sin^3\theta}$

$$\Rightarrow \quad g'(\theta) = e^{\cos^3\theta - \sin^3\theta}\left[3\cos^2\theta(-\sin\theta) - 3\sin^2\theta\cos\theta\right] = 0$$

$$\Rightarrow \quad \sin\theta = 0 \text{ or } \cos\theta = 0 \text{ or } \sin\theta + \cos\theta = 0.$$

If $\sin\theta = 0$ then $\theta = 0$ or $\theta = \pi$.

If $\cos\theta = 0$ then $\theta = \frac{\pi}{2}$ or $\theta = -\frac{\pi}{2}$.

If $\sin\theta + \cos\theta = 0$ then $\theta = \frac{3\pi}{4}$ or $\theta = -\frac{\pi}{4}$.

Let's make a table and find f at all those candidate points:

θ	(x,y)	$f(x,y)$
	$(0,0)$	1
0	$(1,0)$	e^1
π	$(-1,0)$	e^{-1}
$\frac{\pi}{2}$	$(0,1)$	e^{-1}
$-\frac{\pi}{2}$	$(0,-1)$	e^1
$\frac{3\pi}{4}$	$\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$	$e^{-1/\sqrt{2}}$
$-\frac{\pi}{4}$	$\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$	$e^{1/\sqrt{2}}$

Maximum is: $f(1,0) = f(0,-1) = e$

Minimum is: $f(0,1) = f(-1,0) = e^{-1}$