Çankaya University
Department of Mathematics
2016 - 2017 Spring Semester

MATH 156 - Calculus for Engineering II

First Midterm Examination

1) Are the following sequences convergent? Explain.

a) $a_n = \dfrac{\cos(n\pi)}{n}$

b) $a_n = \left(1 - \dfrac{2}{n}\right)^n$

c) $a_n = \dfrac{1 \cdot 3 \cdot 5 \cdots (2n+1)}{(2n+1)!}$

2) Are the following series convergent or divergent? Explain.

a) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n}$

b) $\displaystyle\sum_{n=1}^{\infty} \left(1 - \frac{2}{n}\right)^n$

c) $\displaystyle\sum_{n=0}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n+1)}{(2n+1)!}$

3) Are the following series convergent or divergent? Explain.

a) $\displaystyle\sum_{n=1}^{\infty} n^3 e^{-n^4}$

b) $\displaystyle\sum_{n=1}^{\infty} \frac{n+2017}{n^5 + \ln n + \sqrt{n}}$

c) $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{(n-1)!\,(n+1)!}$

4) Are the following series absolutely convergent, conditionally convergent or divergent? Explain.

a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\, 4^n}{7^n\, n^3}$

b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1}$

5) Find the radius and interval of convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{(x-2017)^n}{n16^n}$.

6) a) Find $\displaystyle\int_0^1 e^{-2x^4}\, dx$ as a series sum.

b) Find the limit $\displaystyle\lim_{x\to 0} \frac{x\left(\cos 2x - 1\right)}{\sin x - x}$.

c) Find the sum of the series $\dfrac{2}{\pi} + \left(\dfrac{2}{\pi}\right)^2 + \left(\dfrac{2}{\pi}\right)^3 + \cdots$

Answers

1) a) $\cos n\pi = \pm 1 \quad \Rightarrow \quad \lim_{n\to\infty} \frac{\cos(n\pi)}{n} = 0$

b) $\lim_{n\to\infty} \left(1 - \frac{2}{n}\right)^n = e^{-2}$

c) $\lim_{n\to\infty} a_n = \frac{1}{2\cdot 4\cdot 6\cdots(2n)} = 0$

2) a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is convergent by Alternating Series Test, because

$a_n = \frac{1}{n} > 0, \quad \lim_{n\to\infty} a_n = 0$ and

$a_{n+1} < a_n.$

b) $\lim_{n\to\infty} \left(1 - \frac{2}{n}\right)^n = e^{-2} \neq 0 \quad \Rightarrow \quad$ divergent by nth term test.

c) Let's use ratio test.

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{1\cdot 3\cdot 5\cdots(2n+1)(2n+3)}{(2n+3)!}}{\frac{1\cdot 3\cdot 5\cdots(2n+1)}{(2n+1)!}}$$

$$= \lim_{n\to\infty} \frac{(2n+1)!(2n+3)}{(2n+3)!}$$

$$= \lim_{n\to\infty} \frac{1}{2n+2}$$

$$= 0$$

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$ therefore the series is convergent by Ratio Test.

3) a) Let $f(x) = x^3 e^{-x^4}$. Clearly, $f(x) > 0$, $f(x)$ is continuous on $[1, \infty)$ and $f(x)$ is decreasing.

Use integral test to obtain: $\int_1^\infty x^3\, e^{-x^4}\, dx$

We can evaluate this using the substitution $u = -x^4,\ du = -4x^3\, dx$ to obtain

$$\int_1^\infty x^3\, e^{-x^4}\, dx = -\frac{e^{-x^4}}{4}\Big|_1^\infty = \frac{e^{-1}}{4}.$$

The integral is convergent (Finite). Therefore the series is convergent by Integral Test.

b) Consider $\sum_{n=1}^{\infty} \frac{1}{n^4}$. We know that this series is convergent by $p-$test, because $4 > 1$.

$$\lim_{n\to\infty} \frac{\dfrac{n+2017}{n^5 + \ln n + \sqrt{n}}}{\dfrac{1}{n^4}} = \lim_{n\to\infty} \frac{n^5 + 2017n^4}{n^5 + \ln n + \sqrt{n}} = 1$$

Therefore $\sum_{n=1}^{\infty} \frac{n+2017}{n^5 + \ln n + \sqrt{n}}$ is also convergent by Limit Comparison Test.

c) $$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\dfrac{(2n+2)!}{(n)!\,(n+2)!}}{\dfrac{(2n)!}{(n-1)!\,(n+1)!}}$$

$$= \lim_{n\to\infty} \frac{(2n+2)!}{(2n)!}\ \frac{(n-1)!}{n!}\ \frac{(n+1)!}{(n+2)!}$$

$$= \lim_{n\to\infty} \frac{(2n+2)(2n+1)}{n(n+2)} = 4 > 1$$

Therefore $\sum_{n=1}^{\infty} \frac{(2n)!}{(n-1)!\,(n+1)!}$ is divergent by Ratio Test.

4) **a)** Consider $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{4^n}{7^n\, n^3}$

$$\lim_{n\to\infty} \left(\frac{4^n}{7^n\, n^3}\right)^{1/n} \quad = \quad \frac{4}{7} \lim_{n\to\infty} n^{-3/n} \quad = \quad \frac{4}{7} \quad < \quad 1$$

Therefore the series is absolutely convergent by Root Test.

b)

- The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1}$ is convergent by Alternating series test, because

$$\frac{1}{2n-1} > 0, \quad \lim_{n\to\infty} \frac{1}{2n-1} = 0$$

and $\frac{1}{2n+1} < \frac{1}{2n-1} \quad \Rightarrow \quad a_{n+1} < a_n$

- We know that $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent (Harmonic series) and $\lim_{n\to\infty} \frac{\frac{1}{2n-1}}{\frac{1}{n}} = \frac{1}{2}$.

  Therefore the series $\sum_{n=1}^{\infty} \frac{1}{2n-1}$ is also divergent by Limit Comparison Test.

- Therefore $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1}$ is conditionally convergent.

5) $$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} \quad = \quad \frac{(x-2017)^{n+1}}{(x-2017)^n} \cdot \frac{n}{n+1} \cdot \frac{16^n}{16^{n+1}} \quad = \quad \frac{x-2017}{16}$$

Therefore the series is convergent for $|x - 2017| < 16$ by Ratio Test.

$2001 < x < 2033$ therefore the center is $x = 2017$ and the radius of convergence is $R = 16$.
Now, let's check endpoints:

$x = 2001 \quad \Rightarrow \quad x - 2017 = -16$

$\sum_{n=1}^{\infty} \frac{(-16)^n}{n16^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ This is convergent by Alternating Series Test.

$x = 2033 \quad \Rightarrow \quad x - 2017 = 16$

$\sum_{n=1}^{\infty} \frac{(16)^n}{n16^n} = \sum_{n=1}^{\infty} \frac{1}{n}$ This is Harmonic series, divergent by Integral Test.

Therefore the interval of convergence is: $x \in [2001, 2033)$ or $2001 \leqslant x < 2033$

6) a) $\int_0^1 e^{-2x^4}\,dx = \int_0^1 \sum_{n=0}^{\infty} \frac{(-2x^4)^n}{n!}\,dx$

$$= \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n\, 2^n\, x^{4n}}{n!}\,dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n\, 2^n\, x^{4n+1}}{(4n+1)\,n!}\Bigg|_0^1$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n\, 2^n}{(4n+1)\,n!} = 1 - \frac{2}{5} + \frac{2}{9} - \frac{4}{39} + \cdots$$

b) Using Taylor series of $\cos 2x$ and $\sin x$ at $x = 0$ we obtain:

$$\lim_{x\to 0} \frac{x\left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \cdots\right) - x}{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) - x}$$

$$= \lim_{x\to 0} \frac{-2x^3 + \frac{16}{24}x^5 - \cdots}{-\frac{x^3}{3!} + \frac{x^5}{5!} - \cdots}$$

$$= 2 \cdot 3!$$

$$= 12$$

c) $\frac{2}{\pi} + \left(\frac{2}{\pi}\right)^2 + \left(\frac{2}{\pi}\right)^3 + \cdots$

$$= \sum_{n=1}^{\infty} \left(\frac{2}{\pi}\right)^n$$

$$= \frac{2}{\pi} \sum_{n=0}^{\infty} \left(\frac{2}{\pi}\right)^n$$

This is geometric series with $r = \frac{2}{\pi} < 1$, therefore it is convergent.

$$= \frac{2}{\pi} \cdot \frac{1}{1 - \frac{2}{\pi}}$$

$$= \frac{2}{\pi - 2}$$