Çankaya University
Department of Mathematics
2016 - 2017 Summer Semester

MATH 156 - Calculus for Engineering II

Final Examination

1) a) Is the series $\sum_{n=0}^{\infty} \frac{n!}{n^2\, 3^{2n+1}}$ convergent or divergent? Explain.

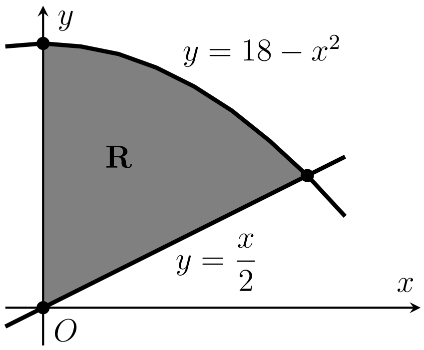
b) Find the sum of the series $\sum_{n=2}^{\infty} \left(\frac{5}{6}\right)^n$.

2) a) Find the plane tangent to the surface $x^2 + 2y^2 + yz = 3$ at the point $P(2, -1, 3)$.

b) Let $f(x, y) = x^2 \ln y + (1 + xy)^4$. Find f_{xy}.

3) Find the maximum of $f(x, y) = yx^4$ where x and y are positive and subject to the constraint $4x + 5y = 25$.

4) a) Evaluate the integral $\iint_R x\, dA$ where R is the region given in figure:



b) Evaluate the integral $\int_0^1 \int_x^{\sqrt{x}} \sqrt{3y^2 - 2y^3}\, dy\, dx$.

5) Evaluate the following integrals:

a) $\int_0^1 \int_{\sqrt{3}\,x}^{\sqrt{4-x^2}} dy\, dx$.

b) $\int_0^2 \int_1^3 \int_2^5 (x^2 - 3y)\, dz\, dy\, dx$.

6) Find the volume of the solid above the cone $z = \sqrt{x^2 + y^2}$ and below the paraboloid $z = 12 - x^2 - y^2$.

Answers

1) a) Ratio Test with $a_n = \dfrac{n!}{n^2 3^{2n+1}} > 0$

$$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\dfrac{(n+1)!}{(n+1)^2\, 3^{2n+3}}}{\dfrac{n!}{n^2\, 3^{2n+1}}} = \lim_{n\to\infty} \frac{n+1}{9}\left(\frac{n}{n+1}\right)^2 = \infty$$

Since $L > 1$ series is divergent by ratio test.

b) $\displaystyle\sum_{n=2}^{\infty}\left(\frac{5}{6}\right)^n = \left(\frac{5}{6}\right)^2 \sum_{n=0}^{\infty}\left(\frac{5}{6}\right)^n = \left(\frac{5}{6}\right)^2 \frac{1}{1-\frac{5}{6}} = \frac{25}{6}$

2) a) $\nabla f = (2x, 4y+z, y) \quad \Rightarrow \quad \nabla f(P) = (4, -1, -1)$

Equation of tangent plane: $4(x-2) - (y+1) - (z-3) = 0 \quad \Rightarrow \quad 4x - y - z = 6$

b) $f_x = 2x \ln y + 4y(1+xy)^3$

$f_{xy} = \dfrac{2x}{y} + 4(1+xy)^3 + 12xy(1+xy)^2$

3)

$$\begin{aligned} \nabla f &= \lambda \nabla g \\ 4yx^3 &= 4\lambda && (1) \\ x^4 &= 5\lambda && (2) \\ 4x + 5y &= 25 && (3) \end{aligned}$$

Then by first and second equations

$\dfrac{x^4}{5} = yx^3 \quad \Rightarrow \quad x^3(x-5y) = 0 \quad \Rightarrow \quad x = 0$ or $x = 5y$

$x = 0 \quad \Rightarrow \quad y = 5$ then first critical point $P_1(0,5)$

$x = 5y \quad \Rightarrow \quad y = 1$ and $x = 5$ then second critical point $P_2(5,1)$

$f(P_1) = 0$ and $f(P_2) = 625$

Then the maximum of f is 625 at the point P_2.

4) a) $18 - x^2 = x/2 \quad \Rightarrow \quad 2x^2 + x - 36 = 0$

$$(2x+9)(x-4) = 0 \quad \Rightarrow \quad x = 4$$

$$\int_0^4 \int_{x/2}^{18-x^2} x \, dy \, dx$$

$$= \int_0^4 \left(18x - x^3 - \frac{x^2}{2}\right) dx$$

$$= \frac{208}{3}$$

b) Reversing the order gives:

$$\int_0^1 \int_{y^2}^{y} \sqrt{3y^2 - 2y^3} \, dx \, dy$$

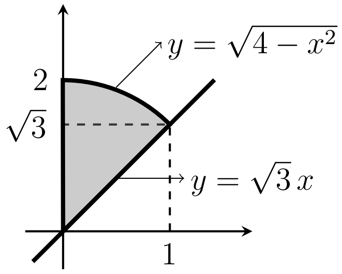
$$= \int_0^1 \left(y^2 - y\right) \sqrt{3y^2 - 2y^3} \, dy$$

$$= \frac{1}{6} \int_0^1 6\left(y^2 - y\right) \sqrt{3y^2 - 2y^3} \, dy$$

$$= \frac{1}{6} \cdot \frac{(3y^2 - 2y^3)^{3/2}}{3/2} \Bigg|_0^1 dy$$

$$= \frac{1}{9}$$

5) a)



$$\int_0^1 \int_{\sqrt{3}\,x}^{\sqrt{4-x^2}} dy\,dx$$

$$= \int_{\pi/3}^{\pi/2} \int_0^2 r dr\, d\theta$$

$$= \int_{\pi/3}^{\pi/2} 2\, d\theta$$

$$= 2\left(\frac{\pi}{2} - \frac{\pi}{3}\right)$$

$$= \frac{\pi}{3}$$

5) b) $\displaystyle\int_0^2 \int_1^3 \int_2^5 \left(x^2 - 3y\right)\, dz\, dy\, dx$

$$= \int_0^2 \int_1^3 3\left(x^2 - 3y\right) dy\, dx$$

$$= \int_0^2 \left(3x^2 y - \frac{9y^2}{2}\right) \Big|_1^3 dx$$

$$= \int_0^2 \left(6x^2 - 36\right) dx = 2x^3 - 36x \Big|_0^2$$

$$= 16 - 72 = -56$$

6) Using polar coordinates, we see that $z = r$ and $z = 12 - r^2$.

Solution of $12 - r^2 = r$ gives $z = 3$ and $r = 3$. Therefore the volume is:

$$V = \int_0^{2\pi} \int_0^3 \int_r^{12-r^2} dz\, r dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^3 \left(12 - r^2 - r\right)\, r dr\, d\theta$$

$$= \int_0^{2\pi} \left(6r^2 - \frac{r^3}{3} - \frac{r^4}{4}\right) \Big|_0^3 d\theta$$

$$= \left(54 - 9 - \frac{81}{4}\right) \int_0^{2\pi} d\theta$$

$$= \frac{99\,\pi}{2}$$