Çankaya University
Department of Mathematics
2016 - 2017 Spring Semester

MATH 156 - Calculus for Engineering II

Final Examination

1) **a)** Is the sequence $a_n = \dfrac{n}{e^n}$ convergent or divergent? Explain.

b) Is the series $\displaystyle\sum_{n=0}^{\infty} \left(\frac{4n}{2n+3}\right)^n$ convergent or divergent? Explain.

c) Find the limit $\displaystyle\lim_{x\to 0} \frac{x\cos x - x}{e^x - 1 - x - \frac{x^2}{2}}$ using the Taylor series expansion of the functions.

2) Is the function $f(x,y) = \begin{cases} \dfrac{x^4(y^3+2)+2y^2}{x^4+y^2} & \text{if } (x,y) \neq (0,0) \\ 2 & \text{if } (x,y) = (0,0) \end{cases}$

continuous at $(0,0)$? Explain.

3) We know that the minimum value of the function $f(x,y) = 2(x-1)^2 + 2(y+2)^2$ subject to the constraint $x + y = c$ is 36. Find possible values of the constant c.

4) Evaluate the integral $\displaystyle\int_0^8 \int_{\sqrt[3]{y}}^2 \cos\left(\frac{\pi x^4}{64}\right)\, dx\, dy.$

5) Evaluate the following integrals:

a) $\displaystyle\iiint_B \left(4x + \frac{5}{z}\right) dV$, where B is the box: $0 \leqslant x \leqslant 1, \quad 0 \leqslant y \leqslant 2, \quad 1 \leqslant z \leqslant 4.$

b) $\displaystyle\int_0^3 \int_0^{3-x} \int_y^3 2\, dz\, dy\, dx.$

6) Find the volume of the solid bounded by the cone $y = \dfrac{1}{\sqrt{2}}\sqrt{x^2+z^2}$ and the paraboloid $y = 3 - x^2 - z^2$.

Answers

1) a) Let $f(x) = \dfrac{x}{e^x}$. Consider $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \dfrac{x}{e^x}$. Using L'Hôpital's Rule, we obtain

$$\lim_{x\to\infty} \frac{1}{e^x} = 0$$

Therefore $\lim\limits_{n\to\infty} a_n = 0$. The sequence is convergent.

b) Using root test we obtain:

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{4n}{2n+3}\right)^n} = \lim_{n\to\infty} \frac{4n}{2n+3} = 2$$

$L = 2 > 1 \quad \Rightarrow \quad$ The series diverges

c) $$\lim_{x\to 0} \frac{x\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) - x}{\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) - 1 - x - \frac{x^2}{2}}$$

$$= \lim_{x\to 0} \frac{-\frac{x^3}{2!} + \frac{x^5}{4!} - \cdots}{\frac{x^3}{3!} + \cdots} = -3$$

2) We can rewrite the function as: $\dfrac{x^4}{x^4+y^2}\, y^3 + 2$.

Clearly $0 \leqslant \dfrac{x^4}{x^4+y^2} \leqslant 1$ therefore $\lim\limits_{y\to 0} \dfrac{x^4}{x^4+y^2}\, y^3 + 2 = 2$

$$\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$$

The limit equals to the function value $\quad \Rightarrow \quad f(x,y)$ is continuous at $(0,0)$

3) Let's use Lagrange multipliers.

$$\nabla f = \lambda \nabla g$$

$$4(x-1) = \lambda, \quad 4(y+2) = \lambda$$

$$\Rightarrow \quad y + 2 = x - 1 \quad \Rightarrow \quad y = x - 3$$

$$x + x - 3 = c \quad \Rightarrow \quad x = \frac{c+3}{2}, \quad y = \frac{c-3}{2}$$

Insert these values in f to obtain:

$$f = 2\left(\frac{c+1}{2}\right)^2 + 2\left(\frac{c+1}{2}\right)^2 = 36$$

$$\left(\frac{c+1}{2}\right)^2 = 9 \quad \Rightarrow \quad \frac{c+1}{2} = \pm 3$$

$c = 5$ OR $c = -7$.

4) We have to reverse the order:

$$\int_0^2 \int_0^{x^3} \cos\left(\frac{\pi x^4}{64}\right) dy\, dx$$

$$= \int_0^2 y \cos\left(\frac{\pi x^4}{64}\right)\Bigg|_0^{x^3} dx$$

$$= \int_0^2 x^3 \cos\left(\frac{\pi x^4}{64}\right) dx$$

Using $u = \dfrac{\pi x^4}{64}, \quad du = \dfrac{\pi x^3\, dx}{16}$ we obtain

$$= \frac{16}{\pi} \sin\left(\frac{\pi x^4}{64}\right)\Bigg|_0^2$$

$$= \frac{16}{\pi} \sin\left(\frac{\pi}{4}\right)$$

$$= \frac{8\sqrt{2}}{\pi}$$

5) a) $\displaystyle\int_0^1 \int_0^2 \int_1^4 \left(4x + \frac{5}{z}\right) dz\, dy\, dx.$

$$= \int_0^1 \int_0^2 (4xz + 5\ln z)\Bigg|_1^4 dy\, dx$$

$$= \int_0^1 (12xy + 5\ln 4y)\Bigg|_0^2 dx$$

$$= 12x^2 + (10\ln 4)x\Bigg|_0^1$$

$$= 12 + 10\ln 4$$

5) b) $\displaystyle\int_0^3 \int_0^{3-x} \int_y^3 2\,dz\,dy\,dx$

$$= \int_0^3 \int_0^{3-x} 6 - 2y\,dy\,dx$$

$$= \int_0^3 6y - y^2 \Big|_0^{3-x} dx$$

$$= \int_0^3 6(3-x) - (3-x)^2\,dx$$

$$= \int_0^3 9 - x^2\,dx$$

$$= 9x - \frac{x^3}{3}\Big|_0^3$$

$$= 27 - 9$$

$$= 18$$

6) Let $r = \sqrt{x^2+z^2}$. At the intersection, $y = 1$ and $r = \sqrt{2}$.

$$V = \int_0^{2\pi} \int_0^{\sqrt{2}} \int_{r/\sqrt{2}}^{3-r^2} dy\,r dr\,d\theta$$

$$= \int_0^{2\pi} \int_0^{\sqrt{2}} \left(3 - r^2 - \frac{r}{\sqrt{2}}\right) r dr\,d\theta$$

$$= \int_0^{2\pi} \left(\frac{3r^2}{2} - \frac{r^4}{4} - \frac{r^3}{3\sqrt{2}}\right)\Big|_0^{\sqrt{2}} d\theta$$

$$= \left(3 - 1 - \frac{2}{3}\right) \int_0^{2\pi} d\theta$$

$$= \frac{4}{3} \cdot 2\pi$$

$$= \frac{8\,\pi}{3}$$